Properties of “Planar Binary (Butchi Number)”

outline

Binary numbers has the rule by the following.

$1_2 + 1_2 = 10_2$

carry leftward

Think the following rule by extension.

$$\begin{array}{r|}1\\\hline\end{array} + \begin{array}{r|}1\\\hline\end{array} = \begin{array}{rr|}0&1\\1&0\\\hline\end{array}$$

carry leftward and upward

We name the number “*planar binary* (*Butchi number*)” by represented in this way.

definition

Butchi number is the numeration system having radix p, q that satisfies $p+q=2$.

It is represented by polynomial in p and q that those coefficients has 1 or 0.

$$\sum_{j=0}^{m}\sum_{i=0}^{n} a_{ij}\, p^i q^j \qquad (a_{ij} \in \{0, 1\})$$

And note the folloing by line up coefficients of polynomial.

$$\begin{array}{ccccc|c}
a_{nm} & \cdots & a_{im} & \cdots & a_{1m} & a_{0m} & q^m\\
\vdots & \ddots & \vdots & & \vdots & \vdots & \\
a_{nj} & \cdots & a_{ij} & \cdots & a_{1j} & a_{0j} & q^j\\
\vdots & & \vdots & \ddots & \vdots & \vdots & \\
a_{n1} & \cdots & a_{i1} & \cdots & a_{11} & a_{01} & q\\
a_{n0} & \cdots & a_{i0} & \cdots & a_{10} & a_{00} & 1\\
\hline
(\ p^n & & p^i & & p & 1\ ) &
\end{array}$$

radix

calculating algorithm of Butchi number expression of natural number

1. set $k \leftarrow 0$
2. calculate according to the following equation from coefficient X at every $p^i q^{k-i}$ with i=0 to k

$$X = \left\lfloor \frac{X}{2} \right\rfloor p + \left\lfloor \frac{X}{2} \right\rfloor q + (X \bmod 2)$$

leftward carry upward carry

3. execute 2. as $k \leftarrow k+1$

For example, 10 is represented by the folloing.

$$\begin{array}{r|}10\\\hline\end{array} = \begin{array}{rr|}0&5\\\mathbf{5}&0\\\hline\end{array} = \begin{array}{rrr|}0&2&\mathbf{5}\\2&1&0\\\hline\end{array} = \begin{array}{rrr|}0&0&2\\0&4&1\\\mathbf{2}&1&0\\\hline\end{array} = \begin{array}{rrrr|}0&0&0&2\\0&1&\mathbf{4}&1\\1&0&1&0\\\hline\end{array} = \begin{array}{rrrr|}0&0&2&\mathbf{2}\\0&3&0&1\\1&0&1&0\\\hline\end{array}$$

$$= \begin{array}{rrrr|}0&0&0&1\\0&0&3&0\\0&\mathbf{3}&0&1\\1&0&1&0\\\hline\end{array} = \begin{array}{rrrr|}0&0&0&1\\0&1&\mathbf{3}&0\\1&1&0&1\\1&0&1&0\\\hline\end{array} = \begin{array}{rrrr|}0&0&1&1\\0&\mathbf{2}&1&0\\1&1&0&1\\1&0&1&0\\\hline\end{array} = \begin{array}{rrrr|}0&1&1&1\\1&0&1&0\\1&1&0&1\\1&0&1&0\\\hline\end{array}$$

addition

$$A = \sum_{j=0}^{m}\sum_{i=0}^{n} a_{ij}\, p^i q^j \qquad B = \sum_{j=0}^{m}\sum_{i=0}^{n} b_{ij}\, p^i q^j$$

$$C = A + B = \sum_{j=0}^{m}\sum_{i=0}^{n} a_{ij}\, p^i q^j + \sum_{j=0}^{m}\sum_{i=0}^{n} b_{ij}\, p^i q^j$$

$$= \sum_{j=0}^{m}\sum_{i=0}^{n} (a_{ij} + b_{ij})\, p^i q^j$$

※Then have to transform coefficients
that greater than 2 to less than 1 by “calculating algorithm”

ex) $\begin{array}{rrr|}0&1&1\\1&0&0\\1&0&1\\\hline\end{array} + \begin{array}{rr|}0&1\\1&1\\\hline\end{array}$ (5 + 3)

$$= \begin{array}{rrr|}0&1&1\\1&0&1\\1&1&\mathbf{2}\\\hline\end{array} = \begin{array}{rrr|}0&1&1\\1&0&2\\1&\mathbf{2}&0\\\hline\end{array} = \begin{array}{rrr|}0&1&1\\1&1&\mathbf{2}\\2&0&0\\\hline\end{array} = \begin{array}{rrr|}0&1&2\\1&2&0\\\mathbf{2}&0&0\\\hline\end{array} = \begin{array}{rrrr|}0&0&1&2\\0&2&\mathbf{2}&0\\1&0&0&0\\\hline\end{array} = \begin{array}{rrrr|}0&0&2&\mathbf{2}\\0&3&0&0\\1&0&0&0\\\hline\end{array} = \begin{array}{rrrr|}0&0&0&1\\0&0&3&0\\0&3&0&0\\1&0&0&0\\\hline\end{array} = \begin{array}{rrrr|}0&0&0&1\\0&1&\mathbf{3}&0\\1&1&0&0\\1&0&0&0\\\hline\end{array} = \begin{array}{rrrr|}0&0&1&1\\0&\mathbf{2}&1&0\\1&1&0&0\\1&0&0&0\\\hline\end{array}$$

$$= \begin{array}{rrrr|}0&1&1&1\\1&0&1&0\\1&1&0&0\\1&0&0&0\\\hline\end{array} (= 8)$$

multiplication

$$A = \sum_{j_1=0}^{m_1}\sum_{i_1=0}^{n_1} a_{i_1 j_1} p^{i_1} q^{j_1} \qquad B = \sum_{j_2=0}^{m_2}\sum_{i_2=0}^{n_2} b_{i_2 j_2} p^{i_2} q^{j_2}$$

$$C = A \times B = \sum_{j_1=0}^{m_1}\sum_{i_1=0}^{n_1} a_{i_1 j_1} p^{i_1} q^{j_1} \times \sum_{j_2=0}^{m_2}\sum_{i_2=0}^{n_2} b_{i_2 j_2} p^{i_2} q^{j_2}$$

$$= \sum_{j_2=0}^{m_2}\sum_{i_2=0}^{n_2} \left\{ \sum_{j_1=0}^{m_1}\sum_{i_1=0}^{n_1} (a_{i_1 j_1} b_{i_2 j_2}) p^{i_1+i_2} q^{j_1+j_2} \right\}$$

※Then have to transform coefficients
that greater than 2 to less than 1 by “calculating algorithm”

ex) $\begin{array}{rrr|}0&1&1\\1&0&0\\1&0&1\\\hline\end{array} \times \begin{array}{rr|}0&1\\1&1\\\hline\end{array}$ (5 ×3) $= \begin{array}{rrr|}0&1&1\\1&0&0\\1&0&1\\\hline\end{array} \times 1 + \begin{array}{rrrr|}0&1&1&0\\1&0&0&0\\1&0&1&0\\\hline\end{array} \times 1 + \begin{array}{rrr|}0&1&1\\1&0&0\\1&0&1\\0&0&0\\\hline\end{array} \times 1 + \begin{array}{rrrr|}0&1&1&0\\1&0&0&0\\1&0&1&0\\0&0&0&0\\\hline\end{array} \times 0$

$$= \begin{array}{rrrr|}0&0&1&1\\0&2&2&1\\1&\mathbf{2}&0&1\\1&1&1&1\\\hline\end{array} = \begin{array}{rrrr|}0&0&1&1\\0&3&\mathbf{2}&1\\2&0&0&1\\1&1&1&1\\\hline\end{array} = \begin{array}{rrrr|}0&0&2&1\\0&4&0&1\\\mathbf{2}&0&0&1\\1&1&1&1\\\hline\end{array} = \begin{array}{rrrrr|}0&0&0&2&1\\0&1&\mathbf{4}&0&1\\1&0&0&0&1\\0&1&1&1&1\\\hline\end{array} = \begin{array}{rrrrr|}0&0&2&\mathbf{2}&1\\0&3&0&0&1\\1&0&0&0&1\\0&1&1&1&1\\\hline\end{array} = \begin{array}{rrrrr|}0&0&0&1&0\\0&0&3&0&1\\0&\mathbf{3}&0&0&1\\1&0&0&0&1\\0&1&1&1&1\\\hline\end{array} = \begin{array}{rrrrr|}0&0&0&1&0\\0&1&\mathbf{3}&0&1\\1&1&0&0&1\\1&0&0&0&1\\0&1&1&1&1\\\hline\end{array} = \begin{array}{rrrrr|}0&0&1&1&0\\0&\mathbf{2}&1&0&1\\1&1&0&0&1\\1&0&0&0&1\\0&1&1&1&1\\\hline\end{array}$$

$$= \begin{array}{rrrrr|}0&1&1&1&0\\1&0&1&0&1\\1&1&0&0&1\\1&0&0&0&1\\0&1&1&1&1\\\hline\end{array} (= 15)$$

identity element

$A + \boxed{0} = A$

$A \times \boxed{1} = A$

associative law

$(A + B) + C = A + (B + C)$

$(A \times B) \times C = A \times (B \times C)$

commutative law

$A + B = B + A$

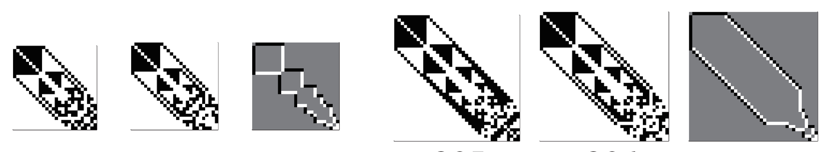
$A \times B = B \times A$

distributive law

$A \times (B + C) = A \times B + A \times C$

increment law

When add 1 to Butchi number, inverting bit makes tho paths.



219 220 395 396

Its algorithm is the folloing.

increment algorithm of Butchi number expression

1-1. if($a_{00} = 0$) $a_{00} \leftarrow 1$ and break.

1-2. if($a_{00} = 1$) $(i_1, j_1) \leftarrow (1, 0)$, $(i_2, j_2) \leftarrow (0, 1)$ and execute the procedure below until $(i_1, j_1) = (i_2, j_2)$

2-1 if($j_1 < j_2$)

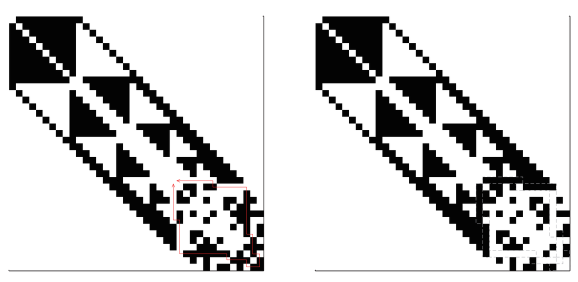
2-1-1 if($a_{i_1 j_1} = 1$) $a_{i_1 j_1} \leftarrow 0$, $i_1 \leftarrow i_1 + 1$

2-1-2 if($a_{i_1 j_1} = 0$) $a_{i_1 j_1} \leftarrow 1$, $j_1 \leftarrow j_1 + 1$

2-2 if($i_2 < i_1$)

2-2-1 if($a_{i_2 j_2} = 1$) $a_{i_2 j_2} \leftarrow 0$, $j_2 \leftarrow j_2 + 1$

2-2-2 if($a_{i_2 j_2} = 0$) $a_{i_2 j_2} \leftarrow 1$, $i_2 \leftarrow i_2 + 1$

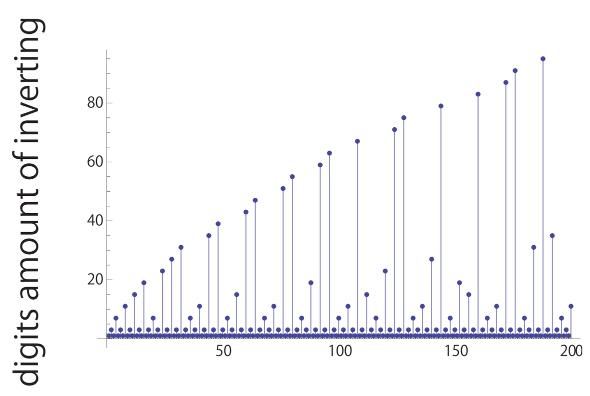


363 364

reduced to its simplest terms,
paths branch like arrows at left figure,
and change directions when bit was changed,
and finally invert bits on paths.

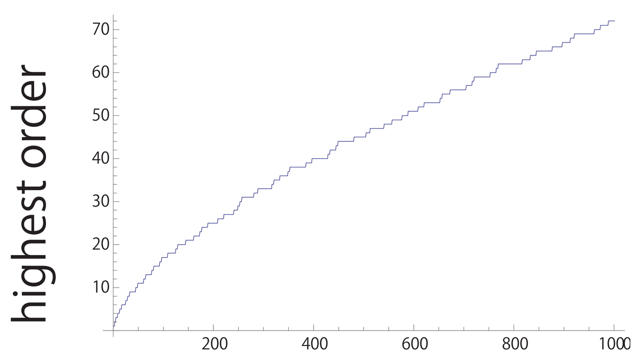
digit amount of change

Plot digits amount of inverting with increment
in Butchi number representation of natural numbers.

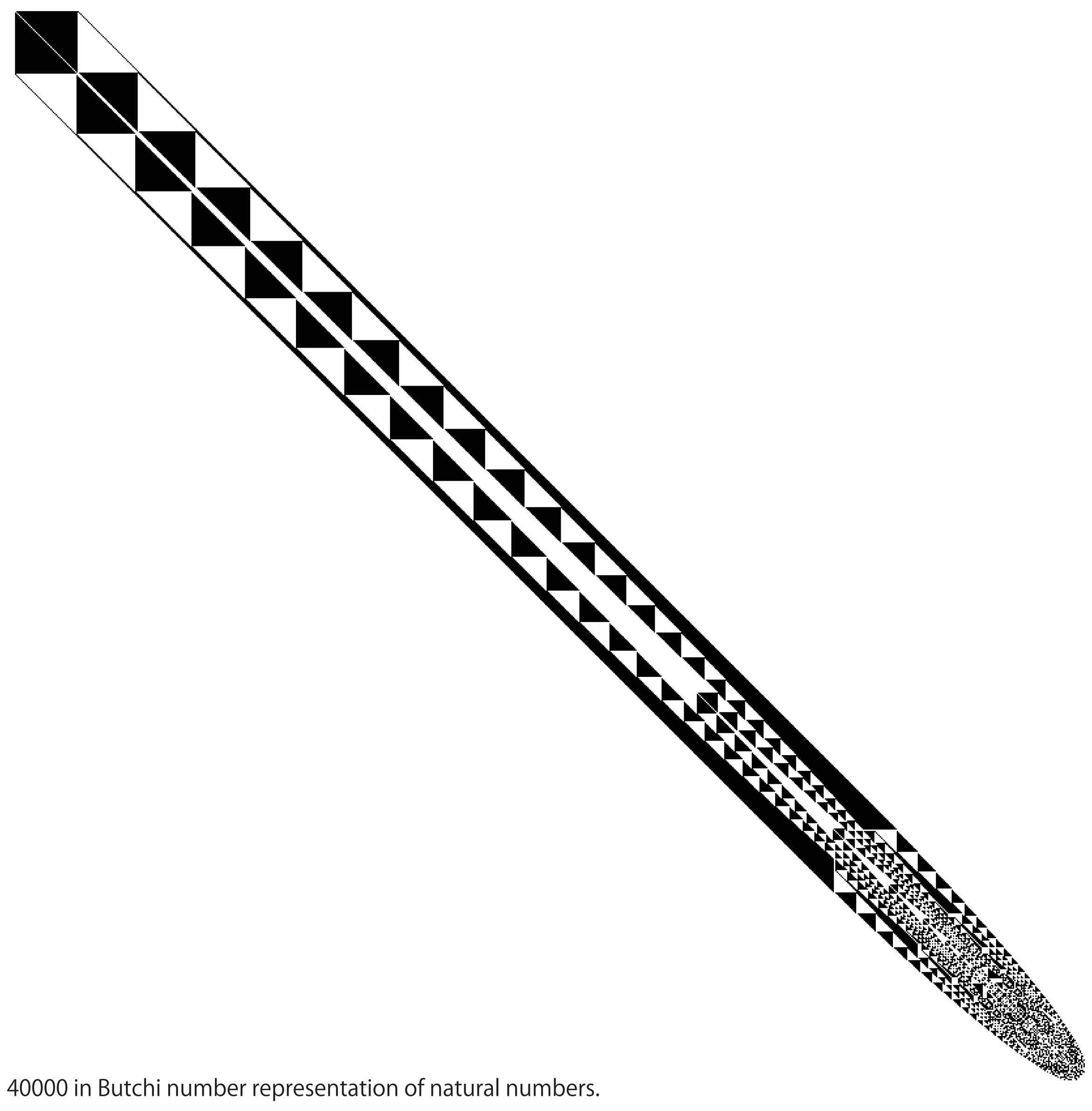


highest order

Plot highest order by polynomial with increment
in Butchi number representation of natural numbers.



large Butchi number



40000 in Butchi number representation of natural numbers.

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